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The path-dependent electrodynamics of systems of bound charges: Lagrangian formulation and canonical relations

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Abstract. The path-dependent Lagrangian is used to derive the energy–momentum tensor of the electromagnetic field and a system of bound charges. The Poisson brackets for the field are defined in a consistent way. Momentum is defined for both field and matter. The Hamiltonian of the combined system is given. The whole description is without any reference to electromagnetic potentials.

1. Introduction

Since the work of Power and Zienau (1959), renewed interest in electrodynamics of systems of bound charges has resulted in a number of papers. These papers were aiming at elimination of electromagnetic potentials from the theory. The connection of these attempts with the path-dependent formulation of electrodynamics (de Witt 1962, Mandelstam 1962), which follows from the paper of Fiutak (1963), was suggested by Woolley (1971), and examined by Fiutak and Engels (1973). The concept of compensating currents (Białynicki-Birula and Białynicka-Birula 1974) has been shown to be closely related to this problem (Fiutak and Żukowski 1978, Woolley 1980). The essential issue of canonical variables and their interpretation has been studied in many papers, from Power and Zienau (1957) to Power and Thirunamachandran (1980) and Rzążewski and Wódkiewicz (1980).

In this work we are interested in electrodynamics of systems of structureless point charges, grouped into stable neutral entities such as atoms and molecules. We are aware of all the complications of a rigorous description of spin particles (compare our attempt (Fiutak and Żukowski 1978)). The whole description presented here is in the domain of classical electrodynamics; however, the quantisation of the formalism can be performed, thanks to the correspondence principle, using the Poisson bracket relations. One is free to choose the very moment of quantisation.

The starting point of this work is the Lagrangian of the electromagnetic field coupled with matter (Fiutak and Żukowski 1978). The interaction term is path dependent because of the polarisation tensor. We take the first pair of the Maxwell equations as our equations of constraints for the electromagnetic fields. This enables us to derive the equations of motion, the energy–momentum tensor and the angular momentum tensor without any recourse to potentials. The energy–momentum tensor for the field is symmetrical from the outset.

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What is perhaps even more important, the present approach enables us to introduce the Poisson brackets again without any reference to field potentials. These Poisson brackets have the same form as the ones already known for phenomenological electrodynamics (Białynicki-Birula and Białynicka-Birula 1974). The set of canonical variables is chosen in such a way that the Poisson brackets of the field variables with the matter variables vanish. Finally, we construct the generators of translations and rotations for any expressions built out of either the field variables or the matter variables. These generators can be called the canonical momentum and canonical angular momentum of the electromagnetic field or the matter. Adding the adjective 'canonical' may be considered unnecessary, but calling them simply 'momenta' and 'angular momenta' may lead to misunderstandings due to the existence of the kinetic momentum of particles. Besides, we want to stress their role in the canonical formalism.

The work presented here is closely related to the problem of derivation of the atomic field equations in the sense of de Groot (1969). We do not want to discuss the whole problem of the derivation of the energy–momentum tensor of the electromagnetic field in ponderable matter (see de Groot and Suttorp 1972). Our intention is to obtain a general description at the microscopic level, leaving the specific problems for future papers.

2. The Lagrangian formulation of bound charges electrodynamics

The Lagrangian of the electromagnetic field and neutral atoms, i.e. stable aggregates of point particles, has been given in Fiutak and Żukowski (1978). It has the following structure:

$$L=\int \mathrm{d}^3x\,\mathscr{L}(x)=\int \mathrm{d}^3x\Big(\mathscr{L}_f(x)+\tfrac{1}{2}m_{\mu\nu}f^{\mu\nu}+\sum_i \mathscr{L}_i(x)\Big) \tag{1}$$

where $\mathscr{L}_f$ is a Lagrangian density of the electromagnetic field, which in the case of the linear Maxwell electrodynamics has the usual form

$$\mathscr{L}_f(x)=-\tfrac{1}{4}f_{\mu\nu}(x)f^{\mu\nu}(x); \tag{2}$$

$\mathscr{L}_i$ is a Lagrangian density describing the particles. The electromagnetic fields satisfy the equations of constraints

$$\partial_\alpha f_{\beta\gamma}+\partial_\gamma f_{\alpha\beta}+\partial_\beta f_{\gamma\alpha}=0, \tag{3}$$

which are usually known as the first pair of the Maxwell equations. Due to the condition (3), and the definition of the polarisation tensor, namely

$$\partial_\beta m^{\alpha\beta}=j^\alpha, \tag{4}$$

where j^α is a current density describing the particles, the transformations of the polarisation tensor given by

$$m^{\alpha\beta}(x)=m'^{\alpha\beta}(x)+\epsilon^{\alpha\beta\mu\nu}\partial_\mu\chi_\nu(x), \tag{5}$$

where χ_ν is an arbitrary differentiable four-field, do not change the equations of motion. These transformations are of the same origin as the usual gauge transformations in the electrodynamics formulated in terms of electromagnetic potentials. One can restrict the possible forms of the polarisation tensors to the so-called path-dependent ones,

$$m^{\alpha\beta}(x)=\sum_k \sum_{i=1}^{n(k)} \int_{\sigma_{ki}} \delta^{(4)}(x-\xi)\,\mathrm{d}\sigma^{\alpha\beta}, \tag{6}$$

where k is the summation index numbering the atoms, and i numbers the particles in the kth atom. The integration is over the dynamical strip σ_{ki} of the particle, which is an arbitrary two-dimensional surface between the trajectory of the kith particle and the trajectory of the centre of the kth atom. For the convenience of the reader we present the main ideas connected with the notion of polarisation tensor in appendix 1. An extensive study of the role of the transformations (5) in the Lagrangian (1) has been given by Healy (1979, 1980, 1977).

Equations of motion, the conservation laws, the definition of the energy–momentum tensor, and the angular momentum tensor are obtained by variations of the action integral

$$W=\int_{\sigma_1}^{\sigma_2} \mathscr{L}(x)\,\mathrm{d}^4x. \tag{7}$$

The integral is over the region of space-time contained between two arbitrary space-like surfaces σ_1 and σ_2. We assume that σ_1 and σ_2 have no common points, and that σ_2 is later than σ_1.

We obtain the equations of motion for the field variables if we choose variations $\delta_0 f_{\alpha\beta}$ of the electromagnetic fields which vanish at the boundary surfaces σ_1 and σ_2. The action principle requires that

$$\delta W=\int_{\sigma_1}^{\sigma_2} \frac{1}{2}\frac{\partial\mathscr{L}}{\partial f_{\mu\nu}}\delta_0 f_{\mu\nu}\,\mathrm{d}^4x=0. \tag{8}$$

However, the variations of the fields $\delta_0 f_{\mu\nu}$ are not independent but subjected to the constraints (3), namely

$$\partial_\alpha\delta_0 f_{\beta\gamma}+\partial_\beta\delta_0 f_{\gamma\alpha}+\partial_\gamma\delta_0 f_{\alpha\beta}=0. \tag{9}$$

The easiest way to take those constraints into account is to present the variations $\delta_0 f_{\alpha\beta}$ in the form

$$\delta_0 f_{\mu\nu}=\partial_\mu\psi_\nu-\partial_\nu\psi_\mu=\partial_{[\mu}\psi_{\nu]} \tag{10}$$

where the ψ_ν are unconstrained, fulfilling only the requirement that $\delta_0 f_{\mu\nu}$ must be infinitesimal. Note that all we need is the form of the variation (10), and this implies that electromagnetic potentials are not introduced. In order to make it more clear and unambiguous, we present in appendix 2 an equivalent approach, based on the path-dependent electrodynamics (de Witt 1962, Mandelstam 1962), which is also free from the concept of electromagnetic potentials.

Since the variation of the action integral is given now by

$$\delta W=-\int_{\sigma_1}^{\sigma_2} \partial_\mu \frac{\partial\mathscr{L}}{\partial f_{\mu\nu}}\psi_\nu\,\mathrm{d}^4x \tag{11}$$

the principle of stationary action implies the following equations of motion for the fields:

$$\partial_\mu\,\partial\mathscr{L}/\partial f_{\mu\nu}=0. \tag{12}$$

One can introduce the traditional notation for $\partial\mathscr{L}/\partial f_{\mu\nu}$, namely $h^{\mu\nu}$. In the case of the Maxwell electrodynamics

$$h^{\mu\nu} = f^{\mu\nu} - m^{\mu\nu}. \tag{13}$$

One can obtain the equations of motion for the particles by varying their trajectories inside the space–time region contained between σ_1 and σ_2. For the ith particle one obtains

$$\delta_0^i W = \int_{\sigma_1}^{\sigma_2} \mathrm{d}^4x(\tfrac{1}{2}\delta_0^i m^{\alpha\beta}(x) f_{\alpha\beta}(x) + \delta_0^i \mathscr{L}_i(x)). \tag{14}$$

The Lagrangian density $\mathscr{L}_i(x)$ for a relativistic point particle is given by

$$\mathscr{L}_i(x) = -m_i c \int_{-\infty}^{+\infty} \mathrm{d}s\, \delta^{(4)}[x - \xi_i(s)], \tag{15}$$

where m_i is the mass of the ith particle, $\xi_i(s)$ its trajectory, and

$$\delta_0^i m^{\alpha\beta}(x) = -e_i \int_{-\infty}^{+\infty} \mathrm{d}s\, \delta[x - \xi_i(s)] \frac{\mathrm{d}\xi_i^{[\alpha}}{\mathrm{d}s} \delta\xi_i^{\beta]}(s)\, \mathrm{d}s. \tag{16}$$

The equations (14)–(16) lead to the well known law of motion for point charges

$$\frac{\mathrm{d}}{\mathrm{d}s} u_\beta^i(s) = \frac{e_i}{m_i} f_{\beta\alpha}[\xi_i(s)] \frac{\mathrm{d}\xi_i^\alpha(s)}{\mathrm{d}s}, \tag{17}$$

where $u^i(s)$ is the four-velocity of the ith particle.

Now as we have the equations of motion governing both fields and particles we can derive the conservation laws. The energy–momentum and angular momentum tensors are defined by the conservation laws resulting from the invariance of the action integral with respect to the Lorentz transformations. Under an infinitesimal Lorentz transformation A, any point x of the space–time changes its coordinates in the following way,

$$(Ax)^\mu = x'^\mu = x^\mu + (\varepsilon^\mu + a^{\mu\nu}x_\nu) = x^\mu + \delta x^\mu, \tag{18}$$

where ε^μ and $a^{\mu\nu}$ are infinitesimals of first order. For any tensor $Z_{\mu\nu}$ we have

$$\delta Z_{\beta\gamma}(x) = Z'_{\beta\gamma}(x) - Z_{\beta\gamma}(x) = -\varepsilon^\mu \partial_\mu Z_{\beta\gamma} - \tfrac{1}{2}a_{\mu\nu}x^{[\mu}\partial^{\nu]}Z_{\beta\gamma} + \tfrac{1}{2}a_{\mu\nu}\delta_{[\gamma}{}^{[\mu}Z_{\beta]}{}^{\nu]}, \tag{19}$$

and finally the action functional is changed by an amount

$$\begin{aligned}\delta W &= \int_{A(\sigma_1)}^{A(\sigma_2)} \mathscr{L}(Ax)\,\mathrm{d}^4x - \int_{\sigma_1}^{\sigma_2} \mathscr{L}(x)\,\mathrm{d}^4x \\ &= \int_{\sigma_1}^{\sigma_2} [\mathscr{L}(Ax) - \mathscr{L}(x)]\,\mathrm{d}^4x + \left(\int_{\sigma_1} - \int_{\sigma_2}\right) \mathrm{d}\sigma_\mu \delta x^\mu \mathscr{L}(x),\end{aligned} \tag{20}$$

where

$$\mathscr{L}(Ax) - \mathscr{L}(x) = \frac{1}{2}\frac{\partial\mathscr{L}}{\partial f_{\mu\nu}} \delta f_{\mu\nu}(x) + \tfrac{1}{2} f_{\mu\nu}(x)\delta m^{\mu\nu}(x) + \sum_i (\mathscr{L}_i(Ax) - \mathscr{L}_i(x)). \tag{21}$$

Using the equations of motion (12) and (17) and the equations of constraints (3), we finally obtain

$$\delta W = F(\sigma_2) - F(\sigma_1), \tag{22}$$

where

$$F(\sigma)=\tfrac{1}{2}M^{\mu\nu}a_{\mu\nu}+P^{\mu}\varepsilon_{\mu}, \tag{23}$$

and the momentum vector P^{μ} and angular momentum tensor are given by

$$P^{\mu}=\int_{\sigma} t^{\mu\nu}\,\mathrm{d}\sigma_{\nu}, \tag{24}$$

$$M^{\mu\nu}=\int_{\sigma}(t^{\rho\mu}x^{\nu}-t^{\rho\nu}x^{\mu})\,\mathrm{d}\sigma_{\rho}. \tag{25}$$

The energy–momentum tensor can be split into two parts, namely

$$t^{\mu\nu}(x)=t_{f}^{\mu\nu}(x)+t_{m}^{\mu\nu}(x), \tag{26}$$

where the field part is given by

$$t_{f}^{\mu\nu}(x)=-(\partial\mathscr{L}_{f}/\partial f_{\mu\gamma})f_{\gamma}{}^{\nu}-g^{\mu\nu}\mathscr{L}_{f}, \tag{27}$$

and the energy–momentum tensor for the particles reads

$$t_{m}^{\mu\nu}(x)=\sum_{i} m_{i}c\int_{-\infty}^{+\infty} u^{i}(s)^{\mu}u^{i}(s)^{\nu}\delta[x-\xi_{i}(s)]\,\mathrm{d}s. \tag{28}$$

Note that $t_{f}^{\mu\nu}$ is symmetric since $\mathscr{L}_{f}$ can be a function of the only possible invariants

$$S=-\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} \qquad \text{and} \qquad P=-\tfrac{1}{4}f_{\mu\nu}\check{f}^{\mu\nu}. \tag{29}$$

In the most general case $t_{f}^{\mu\nu}$ is equal to

$$t_{f}^{\mu\nu}=\frac{\partial\mathscr{L}_{f}}{\partial S}f^{\mu\lambda}f_{\lambda}{}^{\nu}+g^{\mu\nu}\left(\frac{\partial\mathscr{L}_{f}}{\partial P}P-\mathscr{L}_{f}\right). \tag{30}$$

We can conclude this section by stressing that the formalism presented here employs only physical fields and leads straight to the symmetric energy–momentum tensor without any intermediate stage of a canonical non-symmetric one. The procedure can be applied to the free field as well as to the interacting one.

3. The Poisson brackets and canonical momenta

The Poisson bracket relations for the coupled fields can be introduced only at two space-like situated points. It is impossible to extend the PB relations for field variables at general points in the four-dimensional continuum, since it would be necessary to know the solutions of the field equations (12) and (17).

The total energy of the interacting system P^{0} is a functional of the fields $f_{\mu\nu}$ and of the positions and velocities of the charges. In the observer's Lorentz frame it is found that

$$\begin{aligned}P^{0}&=\int \mathrm{d}^{3}\boldsymbol{x}\,t^{00}(\boldsymbol{x})\\&=\int \mathrm{d}^{3}\boldsymbol{x}[(\boldsymbol{d}(\boldsymbol{x})-\boldsymbol{p}(\boldsymbol{x}))\cdot\boldsymbol{e}(\boldsymbol{x})-\mathscr{L}_{f}(\boldsymbol{x})]+\sum_{k}\sum_{i=1}^{m(k)}(c^{2}\boldsymbol{\pi}_{ki}^{2}+m_{ki}^{2}c^{4})^{1/2},\end{aligned} \tag{31}$$

where $\boldsymbol{\pi}_{ki}$ is the kinetic momentum of the kith particle, and

$$\boldsymbol{d} = (h^{01}, h^{02}, h^{03}) = \partial \mathscr{L}/\partial \boldsymbol{e}, \tag{32a}$$

$$\boldsymbol{e} = (f^{01}, f^{02}, f^{03}), \tag{32b}$$

$$\boldsymbol{b} = (f^{23}, f^{31}, f^{12}), \tag{32c}$$

$$\boldsymbol{p} = (m^{10}, m^{20}, m^{30}). \tag{32d}$$

The formula (31) can be rearranged into an equivalent form,

$$P^0 = \int \mathrm{d}^3\boldsymbol{x} \frac{\partial \mathscr{L}}{\partial \boldsymbol{e}} \cdot \boldsymbol{e} + \sum_k \sum_{i=1}^{n(k)} \boldsymbol{p}_{ki} \cdot \dot{\boldsymbol{q}}_{ki} - \int \mathrm{d}^3\boldsymbol{x}\, \mathscr{L}(\boldsymbol{x}), \tag{33}$$

where the canonical momentum of the kith particle is defined as

$$\boldsymbol{p}_{ki} = \partial L/\partial \boldsymbol{q}_{ki}. \tag{33a}$$

The total energy expressed in terms of $\boldsymbol{d}$ and $\boldsymbol{b}$, $\boldsymbol{p}_{ki}$ and $\boldsymbol{q}_{ki}$ is, as we shall see, the Hamiltonian of the system (i.e. the generator of the time translations)

$$H = P^0[\boldsymbol{d}, \boldsymbol{b}; \boldsymbol{p}_{ki}, \boldsymbol{q}_{ki}]. \tag{34}$$

Taking the variation of (33) and using the definition of $\boldsymbol{d}$, we obtain

$$\delta H = \int \mathrm{d}^3\boldsymbol{x}\left(\delta \boldsymbol{d} \cdot \boldsymbol{e} - \frac{\partial \mathscr{L}}{\partial \boldsymbol{b}} \cdot \delta \boldsymbol{b}\right) + \sum \sum \left(\delta \boldsymbol{p}_{ki} \cdot \dot{\boldsymbol{q}}_{ki} - \frac{\partial L}{\partial \boldsymbol{q}_{ki}} \cdot \delta \boldsymbol{q}_{ki}\right). \tag{35}$$

The variations $\delta \boldsymbol{d}$ and $\delta \boldsymbol{b}$ are not unconstrained. The latter one is governed by (9), and $\delta \boldsymbol{d}$ must fulfil the requirement

$$\operatorname{div} \delta \boldsymbol{d}(x) = 0 \tag{36}$$

(due to the equations of motion derived in the previous section). Once more we can define $\delta \boldsymbol{d}(\boldsymbol{x})$ to be

$$\delta \boldsymbol{d} = \nabla \times \boldsymbol{\varphi}(\boldsymbol{x}) \tag{37}$$

where the $\boldsymbol{\varphi}$ are arbitrary in the same sense as the ψ_ν. The new and the old constraints and the equations of motion enable us to transform δH into

$$\delta H = \int \mathrm{d}^3\boldsymbol{x}\, (\dot{\boldsymbol{d}} \cdot \boldsymbol{\psi} - \dot{\boldsymbol{b}} \cdot \boldsymbol{\varphi}) + \sum_k \sum_{i=1}^{n(k)} (\delta \boldsymbol{p}_{ki} \cdot \dot{\boldsymbol{q}}_{ki} - \dot{\boldsymbol{p}}_{ki} \cdot \delta \boldsymbol{q}_{ki}). \tag{38}$$

On the other hand

$$\delta H = \int \mathrm{d}^3x \left(\frac{\delta H}{\delta \boldsymbol{d}} \cdot \delta \boldsymbol{d} + \frac{\delta H}{\delta \boldsymbol{b}} \cdot \delta \boldsymbol{b}\right) + \sum_k \sum_{i=1}^{n(k)} \left(\frac{\partial H}{\partial \boldsymbol{p}_{ki}} \cdot \delta \boldsymbol{p}_{ki} + \frac{\partial H}{\partial \boldsymbol{q}_{ki}} \cdot \delta \boldsymbol{q}_{ki}\right). \tag{39}$$

Using once more the constraints (9) and (36), and comparing the result with (38), we obtain

$$\int \mathrm{d}^3x \left[\left(\dot{\boldsymbol{b}} + \nabla \times \frac{\delta H}{\delta \boldsymbol{d}}\right) \cdot \boldsymbol{\varphi} + \left(\nabla \times \frac{\delta H}{\delta \boldsymbol{b}} - \dot{\boldsymbol{d}}\right) \cdot \boldsymbol{\psi}\right]$$
$$= \sum_{ki} \left[\left(\dot{\boldsymbol{q}}_{ki} - \frac{\partial H}{\partial \boldsymbol{p}_{ki}}\right) \cdot \delta \boldsymbol{p}_{ki} - \left(\dot{\boldsymbol{p}}_{ki} + \frac{\partial H}{\partial \boldsymbol{q}_{ki}}\right) \cdot \delta \boldsymbol{q}_{ki}\right] = 0.$$

The following Hamilton's equations are implied by the last formula:

$$\dot{\boldsymbol{b}} = -\nabla\times \delta H/\delta \boldsymbol{d}, \tag{40a}$$

$$\dot{\boldsymbol{d}} = \nabla\times \delta H/\delta \boldsymbol{b}, \tag{40b}$$

$$\dot{\boldsymbol{q}}_{ki} = \partial H/\partial \boldsymbol{p}_{ki}, \tag{40c}$$

$$\dot{\boldsymbol{p}}_{ki} = -\partial H/\partial \boldsymbol{q}_{ki}. \tag{40d}$$

If we introduce the Poisson brackets for the system defined as

$$\{\mathscr{F}[\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{q}_{ki}, \boldsymbol{p}_{ki}], \mathscr{G}[\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{q}_{ki}, \boldsymbol{p}_{ki}]\} = \int \mathrm{d}^3\boldsymbol{x}\left(\frac{\delta\mathscr{F}}{\delta\boldsymbol{d}}\cdot\nabla\times\frac{\delta\mathscr{G}}{\delta\boldsymbol{b}} - \frac{\delta\mathscr{G}}{\delta\boldsymbol{d}}\cdot\nabla\times\frac{\delta\mathscr{F}}{\delta\boldsymbol{b}}\right) + \sum_k\sum_{i=1}^{n(k)}\left(\frac{\partial\mathscr{F}}{\partial\boldsymbol{q}_{ki}}\cdot\frac{\partial\mathscr{G}}{\partial\boldsymbol{p}_{ki}} - \frac{\partial\mathscr{G}}{\partial\boldsymbol{q}_{ki}}\cdot\frac{\partial\mathscr{F}}{\partial\boldsymbol{p}_{ki}}\right), \tag{41}$$

the equations (40) can be treated as examples of a general law of dynamical evolution of any observable

$$\{\mathscr{F}, H\} = \dot{\mathscr{F}}.$$

This definition of the Poisson brackets is well justified, because of the following facts. The usual association of P^ν and $M^{\mu\nu}$ (expressed by the canonical variables) with the infinitesimal Lorentz transformations is possible using (41) as the definition of the PB. For any field f_μ an infinitesimal Lorentz transformation changes it by

$$\delta f_\mu = \{f_\mu, F(\sigma)\}$$

where f_μ and $F(\sigma)$ are expressed in terms of the canonical variables in the Lorentz frame associated with the hyperplane σ. The Poisson brackets (41) together with the definitions of P^μ and $M^{\mu\nu}$ introduce the correct structure of the Lorentz group from the point of view of the Lie algebra formalism.

Using the general definition, one can easily derive the basic Poisson bracket relations for the canonical variables describing the fields

$$\{b_i(x), d_j(y)\} = \epsilon_{ijk}\partial_k\delta(x-y),$$

$$\{d_i(x), d_j(y)\} = \{b_j(x), b_i(x)\} = \{d_i(x), (q_{ki})_j\} = \{d_i(x), (p_{ki})_j\} = \{b_i(x), (q_{ki})_j\} = \{b_i(x), (p_{ki})_j\} = 0. \tag{42}$$

Applying the correspondence principle, we can use these relations for quantisation of the whole system.

The Poisson brackets for the variables describing the particles are introduced using (41), and are as follows:

$$\{(q_{ki})_r, (q_{k'i'})_s\} = \{(p_{ki})_r, (p_{k'i'})_s\} = 0, \tag{43a}$$

$$\{(q_{ki})_r, (p_{k'i'})_s\} = \delta_{kk'}\delta_{ii'}\delta_{rs}, \tag{43b}$$

where ki and $k'i'$ number the particles, and r and s the coordinates. For the path-dependent polarisation tensor, based on straight lines, the canonical momentum

p_{ki} of the $k i$th particle is (in the special reference frame—see appendix 1)

$$p_{ki} = \pi_{ki} - \left(\frac{e_{ki}}{c}\right) \int_0^1 \mathrm{d}l\, l(q_{ki} - R_k) \times b[R_k + l(q_{ki} - R_k)]$$
$$+ w_{ki} \sum_{j=1}^{n(k)} \left(\frac{e_{kj}}{c}\right) \int_0^1 (1-l)(q_{kj} - R_k) \times b(R_k + l(q_{kj} - R_k))\, \mathrm{d}l \tag{44}$$

where w_{ki} are arbitrary weights, and

$$R_k = \sum_{i=1}^{n(k)} w_{ki} q_{ki}$$

and represents the central point of the kth atom. The central point fulfils the requirement

$$\left\{(R_k)_r, \left(\sum_{i=1}^{n(k)} p_{ki}\right)_s\right\} = \delta_{rs} \tag{45}$$

which means that R_k must be a canonical conjugate of the total canonical momentum of the atom. The total canonical momentum of the atom is the generator of space translations for the atomic variables. The last sentence will be understood as the definition of this quantity.

The notion of the total canonical momentum can be used in the analysis of the field variables. The relation between the total kinetic and canonical momentum of an atom can be written as

$$\pi_k = \sum_{i=1}^{n(k)} \pi_{ki} = P_k + \int \mathrm{d}^3 x\, p_k(x) \times b(x) \tag{46}$$

where $p_k(x)$ is the polarisation of the kth atom, and P_k is the total canonical momentum of the atom. The formula holds for arbitrary R_k fulfilling (45). The polarisation in (46) need not be described by straight lines, since the formula is invariant with respect to arbitrary modifications of the polarisation tensor which satisfy

$$p'(x) = p(x) + \nabla \times C(x), \tag{47a}$$

$$m'(x) = m(x) - \mathrm{d}C(x)/\mathrm{d}t, \tag{47b}$$

where $C(x, t)$ is a function vanishing outside a certain volume containing the atom. The total momentum of the electromagnetic fields and particles can now be decomposed into

$$P = P_f + P_m = \int d(x) \times b(x)\, \mathrm{d}^3 x + \sum_k P_k \tag{48}$$

where the summation is over all atoms. The field part of the splitting (48) will be called the total canonical momentum of the electromagnetic fields.

It should be stressed that the splitting (48) is possible only for definitions of the polarisation tensor that use central points satisfying (45). All the other definitions of the polarisation tensor, e.g. with a fixed central point, presented by Woolley (1975), or the independently moving one used by Healy (1977), break the relations (46) and (48). In our view the fixed point approach to the polarisation tensor is nothing more than using a special kind of potential.

The total canonical momentum of the electromagnetic fields, which we shall denote by P_f, has the property

$$\{(P_f)_i, d_k(x)\} = \partial_i d_k(x), \tag{49a}$$

$$\{(P_f)_i, b_k(x)\} = \partial_i b_k(x), \tag{49b}$$

which justifies its name.

By analogy one can introduce the total canonical angular momentum of the fields

$$\boldsymbol{M}_f = \int \mathrm{d}^3\boldsymbol{x}[\boldsymbol{x} \times (\boldsymbol{d}(\boldsymbol{x}) \times \boldsymbol{b}(\boldsymbol{x}))] \tag{50}$$

and of the particles

$$\boldsymbol{M}_m = \sum_k \dot{\boldsymbol{R}}_k \times \boldsymbol{P}_k + \sum_k \sum_{i=1}^{n(k)} (\boldsymbol{q}_{ki} - \boldsymbol{R}_k) \times \boldsymbol{p}_{ki}. \tag{51}$$

Thus the total angular momentum can be split into the two canonical parts. The total canonical angular momenta serve as generators of rotations for the canonical variables

$$\{(M_f)_i, d_r(x)\} = -\epsilon_{ilr} d_l(x) + \epsilon_{iln} x_l \partial_n d_r(x),$$

$$\{(M_f)_i, b_r(x)\} = -\epsilon_{ilr} b_l(x) + \epsilon_{iln} x_l \partial_n b_r(x),$$

$$\{(M_m)_l, (q_{ki})_r\} = -\epsilon_{lnr} (q_{ki})_n,$$

$$\{(M_m)_l, (q_{ki})_r\} = -\epsilon_{lnr} (p_{ki})_n.$$

4. Concluding remarks

The formalism presented here is well suited for any models of electrodynamics. We have given the Lagrange and Hamilton formalism for the fields interacting with neutral atoms. The Poisson bracket relations enable us to quantise the electromagnetic fields without introducing potentials.

The $\boldsymbol{b}$ and $\boldsymbol{d}$ fields are the canonical variables describing the electromagnetic field. The canonical momentum and the canonical angular momentum are functionals of these fields. On the other hand the path-independent fields $\boldsymbol{b}$ and $\boldsymbol{e}$ serve as the Lagrangian variables.

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Appendix 1. Polarisation tensor

The current density of a bound system of point charges is given by

$$j^\alpha(z) = \sum_k \sum_{i=1}^{n(k)} e_{ki} \int_{P_{ki}} \delta^{(4)}(z - \xi_{ki})\, \mathrm{d}\xi^\alpha_{ki} \tag{A1.1}$$

where e_{ki} is the electric charge of the kith particle and P_{ki} its trajectory and $n(k)$ is the number of particles in the kth atom. One can introduce a privileged 'central' trajectory with which the motion of the kth aggregate of the charged particles as a whole can be identified. The central trajectory depends on the positions of the constituent particles in a certain reference frame. The parametrisations of all trajectories associated with the kth atom are 'synchronised' in this reference frame, i.e.

$$\xi^0_{ki}(\lambda) = R^0_k(\lambda) \tag{A1.2}$$

in the specified frame; $R^\mu_k(\lambda)$ represents the 'central' trajectory.

For such a system the polarisation tensor is given by (6), and can be called a path-dependent one since in any reference frame the polarisation and magnetisation vectors are in fact

$$\boldsymbol{p}(\boldsymbol{r}, t) = \sum_k \sum_{i=1}^{n(k)} e_{ki} \int_{C_{ki}} \mathrm{d}\boldsymbol{\xi}\, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{\xi}), \tag{A1.3}$$

$$\boldsymbol{m}(\boldsymbol{r}, t) = \sum_k \sum_{i=1}^{n(k)} \frac{e_{ki}}{c} \int_{C_{ki}} \mathrm{d}\boldsymbol{\xi} \times \dot{\boldsymbol{\xi}} \delta^{(3)}(\boldsymbol{r} - \boldsymbol{\xi}), \tag{A1.4}$$

where the path C_{ki} is a result of the intersection of the dynamical strip σ_{ki} and the space-like plane associated with the reference frame.

The path-dependent polarisation tensor is associated with a family of compensating currents of the form

$$\sum_{i=1}^{n(k)} j^\mu_c(z; z^i_1, z^i_2) = \sum_{i=1}^{n(k)} \int \mathrm{d}^4x\, a^\mu(z, x, R(x)) \partial_\alpha j^\alpha(x; z^i_1, z^i_2),$$

where

$$a^\mu(z, x, R(x)) = \int_x^{R(x)} \delta^{(4)}(z - \xi)\, \mathrm{d}\xi^\mu,$$

and $R(\xi_{ki}(\lambda)) = R_k(\lambda)$. The non-conserved current $j^\alpha(x; z^i_1, z^i_2)$ satisfies the equation

$$\sum_{i=1}^{n(k)} \partial_\alpha j^\alpha(x; z^i_1, z^i_2) = \partial_\alpha \left(\sum_{i=1}^{n(k)} \int_{z^i_1}^{z^i_2} \delta^{(4)}(z - \xi_{ki})\, \mathrm{d}\xi^\alpha_{ki} \right)$$

$$= \sum_{i=1}^{n(k)} (\delta^{(4)}(x - z^i_1) - \delta^{(4)}(x - z^i_2)).$$

Appendix 2. An alternative way of deriving the equations of motion

It is easy to check that one can transform unconstrained variations of the fields $\delta f_{\alpha\beta}$ into constrained ones $\delta_c f_{\alpha\beta}$ (by constrained variations we understand such that satisfy (9)). One can find that

$$\delta_c f_{\mu\nu}(x) = \partial_\mu \int_\infty^x \delta f_{\alpha\beta}(z) \frac{\partial z^\alpha}{\partial x^\nu} \mathrm{d}z^\beta - \partial_\nu \int_\infty^x \delta f_{\alpha\beta}(z) \frac{\partial z^\alpha}{\partial x^\mu} \mathrm{d}z^\beta. \tag{A2.1}$$

The integration in (A2.1) is along an arbitrary space-like path leading towards the point x from infinity.

The variations $\delta_0 f_{\alpha\beta}$ are equal to zero at the boundary surfaces σ_1 and σ_2 in δW. Hence

$$\begin{aligned}\delta W &= \int_{\sigma_1}^{\sigma_2} \frac{1}{2}\frac{\partial \mathscr{L}}{\partial f_{\mu\nu}} \delta_c f_{\mu\nu}(x)\,\mathrm{d}^4x \\ &= -\int_{\sigma_1}^{\sigma_2} \partial_\nu \frac{\partial \mathscr{L}}{\partial f_{\mu\nu}}(x) \int_\infty^x \delta_0 f_{\alpha\beta}(z) \frac{\partial z^\alpha}{\partial x^\mu}\,\mathrm{d}z^\beta \\ &\quad + \left(\int_{\sigma_1} - \int_{\sigma_2}\right) \mathrm{d}\sigma_\nu \frac{\partial \mathscr{L}}{\partial f_{\mu\nu}}(x) \int_\infty^x \delta_0 f_{\alpha\beta}(z) \frac{\partial z^\alpha}{\partial x^\mu}\,\mathrm{d}z^\beta. \end{aligned} \qquad \text{(A2.2)}$$

One can choose the paths in the surface integrals to be inside the respective surfaces, and thus the last term in (A2.2) vanishes. The equations of motion (12) follow from (A2.2) since $\delta_0 f_{\alpha\beta}$ in the volume integral are arbitrary.

References

Białynicki-Birula I and Białynicka-Birula Z 1974 *Quantum Electrodynamics* (Warszawa: PWN; Oxford: Pergamon)
Fiutak J 1963 *Can. J. Phys.* **41** 12
Fiutak J and Engels Z 1973 *Acta Phys. Pol.* A **44** 795
Fiutak J and Żukowski M 1978 *Acta Phys. Pol.* A **54** 533
de Groot S R 1969 *Maxwell Equations* (Amsterdam: North-Holland)
de Groot S R and Suttorp L G 1972 *Foundations of Electrodynamics* (Amsterdam: North-Holland)
Healy W P 1977 *Proc. R. Soc.* A **358** 367
—— 1979 *Phys. Rev.* A **19** 2353
—— 1980 *J. Phys. A: Math. Gen.* **13** 2383
Mandelstam S 1962 *Ann. Phys., NY* **19** 1
Power S A and Thirunamachandran T 1980 *Proc. R. Soc.* A **372** 265
Power S A and Zienau S 1957 *Nuovo Cimento* **6** 7
—— 1959 *Phil. Trans. R. Soc.* A **251** 427
Rzążewski K and Wódkiewicz K 1980 *Ann. Phys., NY* **130** 1
de Witt B S 1962 *Phys. Rev.* **125** 2189
Woolley R G 1971 *Proc. R. Soc.* A **321** 557
—— 1975 *Adv. Chem. Phys.* **33** 153
—— 1980 *J. Phys. A: Math. Gen.* **13** 2795